

ON NONLOCAL NONLINEAR ELLIPTIC PROBLEM WITH THE FRACTIONAL LAPLACIAN

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ABSTRACT. In this paper, we study a nonlocal elliptic problem with the fractional Laplacian on R^n . We show that the problem has infinite positive solutions in $C^\tau(R^n) \cap H_{loc}^\alpha(R^n)$. Moreover each of these solutions tends to some positive constant limit at infinity.

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1. INTRODUCTION

We prove the following result.

Theorem 1. *Assume $0 < \alpha < n$ and $p > 1$. Let $\omega : R_+ \rightarrow R_+$ be the monotone non-increasing function such that*

$$(1) \quad \int_1^\infty \frac{\omega(r)}{r} dr = A < \infty$$

Then there is a positive constant θ such that for smooth functions $k(x)$ and $K(x)$ with $|K(x)| \leq \theta\omega(|x|)(1 + |x|)^{-\tau}$ and $0 \leq k(x) \leq \theta\omega(|x|)(1 + |x|)^{-\tau}$ on R^n for some $\tau \geq \alpha$, the problem

$$(2) \quad (-\Delta)^{\alpha/2} u + k(x)u = K(x)u^p, \quad \text{in } R^n$$

has infinite positive solutions in $C^\tau(R^n) \cap H_{loc}^\alpha(R^n)$. Moreover, each of these solutions tends to some positive constant limit at infinity.

We prove the above result by using the Perron method and similar argument as in Lin's work [3]. The Perron method is based on the maximum principle developed in [1]. Similar argument had been carried out in [4] for nonlinear subelliptic equations on R^n .

We denote by C the uniform constants, which may vary in different inequalities or formulae.

Denote by $B_1(0)$ the unit ball in R^n .

We present the potential analysis as in [2] in section 2 and we prove the main result in section 3.

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2. PRELIMINARY

Let $\omega : R^n \rightarrow R$ be a radially symmetric monotone non-increasing function. Let $f : R^n \rightarrow R$ be a locally Holder continuous function with the decay growth as below:

$$(3) \quad |f(x)| \leq C\omega(|x|)|x|^{-\tau}$$

where $C > 0$ and $\tau > 2\alpha$ are uniform constants.

Lemma 2. *Define*

$$w(x) = c_n \int_{R^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy,$$

where $c_n = [n(n-2)|B_1(0)|]^{-1}$. Then $w(x)$ is well-defined and near ∞ we have

$$(4) \quad |w(x)| \leq \begin{cases} C|x|^{\alpha-n}\omega(|x|) & \text{if } \tau > n \\ C\omega(|x|) \log |x| & \text{if } \tau = n \\ C\omega(|x|)|x|^{\alpha-\tau} & \text{if } \alpha \leq \tau \end{cases}$$

where $c_n = 1/n(n-2)|B_1(0)|$.

Proof. Clearly we have

$$|w(x)| \leq C \int_{R^n} \frac{\omega(|x|)}{|x-y|^{n-\alpha}(1+|y|^\tau)} dy.$$

We now divide the whole space R^n into three parts:

$$D_1 = \{y \in R^n; |y-x| \leq |x|/2\},$$

$$D_2 = \{y \in R^n; |x|/2 \leq |y-x| \leq 2|x|\},$$

and

$$D_3 = \{y \in R^n; |y-x| \geq 2|x|\}.$$

Set, for $j = 1, 2, 3$,

$$I_j = \int_{D_j} \frac{\omega(|x|)}{|x-y|^{n-\alpha}(1+|y|^\tau)} dy.$$

Then

$$|w(x)| \leq C(I_1 + I_2 + I_3).$$

One can show as in [2] that $w(x)$ has the upper bounds as in (4). \square

We now assume that $\omega(x)$ satisfies (1).

Lemma 3. *Assume $f \geq 0$ on R^n and $f(x) \geq C\omega(|x|)|x|^{-\tau}$ for some $\tau \geq \alpha$ and $C > 0$. The Riesz potential $w(x)$ defined in Lemma 1 has the following lower bounds at ∞ :*

$$(5) \quad w(x) \geq \begin{cases} C|x|^{\alpha-n}\omega(|x|) & \text{if } \tau > n \\ C\omega(|x|) \log |x| & \text{if } \tau = n \\ C\omega(|x|)|x|^{\alpha-\tau} & \text{if } \alpha \leq \tau \end{cases}$$

Proof. As in the proof above, we have

$$w(x) = \left(\int_{D_1} + \int_{D_2} + \int_{D_3} \right) \frac{c_n f(y)}{|x-y|^{n-\alpha}} dy \geq \int_{D_2} \frac{c_n f(y)}{|x-y|^{n-\alpha}} dy := J.$$

One can show as in [2] that J has the lower bounds as in (5).

□

3. PROOF OF THEOREM 1

The argument presented below is similar to Lin's work [3]. We just do it briefly.

Take some constants $\theta_1 > 0$ and $0 < a < 1$. We first define the function $U_a(x)$ on R^n by solving the nonlocal equation

$$(-\Delta)^{\alpha/2} u = -\frac{C\omega(x)}{(1+|x|)^\tau}, \quad \text{on } R^n$$

for $C \in (0, \theta_1)$ with $0 < U_a(x) \leq a$ and $U_a(x) \rightarrow a$ at infinity. We can verify that U_a is the lower solution to (2).

Then we define the function $U^a(x)$ on R^n by solving the nonlocal equation

$$(-\Delta)^{\alpha/2} u = \frac{C\omega(x)}{(1+|x|)^\tau}, \quad \text{on } R^n$$

for the same $C > 0$ with $a \leq U^a(x) < 1$ and $U^a(x) \rightarrow a$ at infinity. We can verify that U^a is the upper solution to (2).

Then we can use the Perron method [1][3] to get the desired solution $u(x)$ to (2) such that $U_a(x) \leq u(x) \leq U^a(x)$ on R^n .

This then completes the proof of Theorem 1.

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